Worcester County Mathematics League

Varsity Meet 1 - October 3, 2018

COACHES' COPY
ROUNDS, ANSWERS, AND SOLUTIONS

Worcester County Mathematics League

Varsity Meet 1 - October 3, 2018

Answer Key

Round 1 - Arithmetic

1. $\frac{19}{20}$ or .95
2. $\frac{5}{3}$ or $1\frac{2}{3}$ or $1.\overline{6}$
3. $\frac{49}{4}$ or $12\frac{1}{4}$ or 12.25

Round 2 - Algebra I

1. 18
2. $(x, y) = \left(\frac{1}{2}, 1\right)$
3. $\frac{60}{13}$ hours or $4\frac{8}{13}$ hours

Round 3 - Set Theory

1. 29 elements
2. 37 seniors
3. 265 integers

Round 4 - Measurement

1. 13 meters
2. 31 units
3. 14 meters

Round 5 - Polynomial Equations

1. $x = 0, 1, -6$
2. $-1 + 2i, -1 - 2i$
3. $3 - 2i, -3 + 2i$

Team Round

1. 129
2. 15
3. 140 times
4. 10 yellow socks
5. $\sqrt{3}$ un^2
6. $-1, \frac{1}{2} + \frac{\sqrt{3}}{2}i, \frac{1}{2} - \frac{\sqrt{3}}{2}i$
7. $10 - \sqrt[3]{500}$ cm
8. $k = 23$
9. 5 blocks

Worcester County Mathematics League
Varsity Meet 1 - October 3, 2018
Round 1 - Arithmetic

All answers must be in simplest exact form in the answer section.

NO CALCULATORS ALLOWED

1. Simplify the expression $\dfrac{4+3\,(15-(-5))}{8-7(24-32)} \div \dfrac{3\cdot 5-5}{11-\frac{3}{2}}$.

2. If $a \ddagger b = \dfrac{a+2b}{2}$ and $a \triangle b = \dfrac{2a-b}{a}$, evaluate $3^{(2\ddagger 1)} \triangle 3$.

3. Let $\boxed{\text{A}}$ be the arithmetic mean of all odd prime numbers less than A, and let $\textcircled{\text{B}}$ be the number of odd prime numbers less than B. Find the value of the following expression.

$$\boxed{\textcircled{\boxed{20}\times\boxed{18}}}$$

ANSWERS

(1 pt) 1. ____________________

(2 pts) 2. ____________________

(3 pts) 3. ____________________

Quaboag, West Boylston, QSC

Worcester County Mathematics League
Varsity Meet 1 - October 3, 2018
Round 2 - Algebra I

All answers must be in simplest exact form in the answer section.
NO CALCULATORS ALLOWED

1. If $a < b$, then $3^2 + 4^2 + 5^2 + 12^2 = a^2 + b^2$ is satisfied by only one pair of positive integers a and b. What is the value of $a + b$?

2. Solve for x and y.

$$\begin{cases} \frac{6}{x} - \frac{1}{y} & = 11 \\ \frac{2}{x} + \frac{1}{y} & = 5 \end{cases}$$

3. Al and Bob, working together, can do a job in 4 hours. Bob and Carl, working together can do it in 5 hours. All three working together can do it in 3 hours. How long will it take Al and Carl, working as usual, to do the job together without Bob?

ANSWERS

(1 pt) 1. ______________________

(2 pts) 2. $(x, y) =$ ______________________

(3 pts) 3. ______________________ hours

Shepherd Hill, Shrewsbury, Doherty

Worcester County Mathematics League

Varsity Meet 1 - October 3, 2018

Round 3 - Set Theory

All answers must be in simplest exact form in the answer section.

NO CALCULATORS ALLOWED

1. Set A contains 17 elements, set B contains 24 elements, and $A \cap B$ contains 12 elements. How many elements are contained in $A \cup B$?

2. The principal of Rosetta Stone High School proudly reported that out of 150 members of the senior class,

   - 75 studied French,
   - 70 studied German,
   - 60 studied Spanish.
   - 40 studied French and German,
   - 30 studied French and Spanish,
   - and 25 studied German and Spanish.

   If three seniors studied all three languages, how many seniors studied none of these languages?

3. How many positive integers less than 1000 have 6, 10, or 15 as factors?

ANSWERS

(1 pt) 1. ________________ elements

(2 pts) 2. ________________ seniors

(3 pts) 3. ________________ integers

Shrewsbury, St. John's, AMSA

Worcester County Mathematics League

Varsity Meet 1 - October 3, 2018

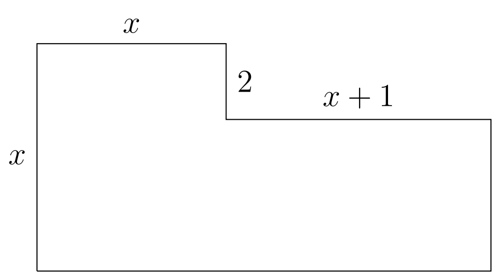
Round 4 - Measurement

All answers must be in simplest exact form in the answer section.

NO CALCULATORS ALLOWED

1. Two vertical poles, 10 meters high and 15 meters high, stand 12 meters apart. A wire is strung tight between the tops of the poles. Find the length of the wire.

2. If the area of the figure below is 64 square units, find the perimeter.



3. A regular square pyramid with equilateral triangles as lateral faces has volume $\frac{1372\sqrt{2}}{3}$ cubic meters. What is the length of a lateral edge?

ANSWERS

(1 pt) 1. __________________ meters

(2 pts) 2. __________________ units

(3 pts) 3. __________________ meters

Bromfield, Quaboag, Shrewsbury

Worcester County Mathematics League

Varsity Meet 1 - October 3, 2018

Round 5 - Polynomial Equations

All answers must be in simplest exact form in the answer section.

NO CALCULATORS ALLOWED

1. Find all values of x which satisfy

$$4^{x^3+5x^2-6x} = 1.$$

2. Given that 3 and $-\frac{1}{2}$ are solutions of

$$2x^4 - x^3 - 3x^2 - 31x - 15 = 0,$$

determine the other roots.

3. Determine the square roots of $5 - 12i$.

ANSWERS

(1 pt) 1. ____________________

(2 pts) 2. ____________________

(3 pts) 3. ____________________

Burncoat, Leicester, Algonquin

Worcester County Mathematics League
Varsity Meet 1 - October 3, 2018
Team Round

All answers must be in simplest exact form in the answer section.
NO CALCULATORS ALLOWED

1. If $x\&y = xy + 1$ and $x\#y = x + y + 1$, what is the value of $4\&[(6\#8)\#(3\&5)]$?

2. If

$$\begin{cases} x + y + z &= 9 \\ x + 2y + 4z &= 15 \\ x + 3y + 9z &= 23 \end{cases}$$

determine the value of xyz.

3. How many times does the digit two appear if you write out the integers between 100 and 300?

4. Paul's sock drawer is filled with 26 socks that are either red, yellow, or blue. The probability that Paul pulls a red sock, a yellow sock, and a blue sock (in that order) is $\frac{1}{40}$. If red socks make up half of his drawer and he has more yellow socks than blue socks, how many yellow socks does Paul have?

5. Regular hexagon ABCDEF is inscribed in circle O whose area is 4π. Find the area of ΔBCD.

6. Find all complex numbers whose negative reciprocal is equal to its square.

7. Lynn has just filled up her cone-shaped water bottle, which has a base radius of 6 cm and a height of 10 cm. She drinks exactly half of her water and sets the bottle back down onto the table base-down. Determine the height of the water level in the bottle.

8. Determine the value of k that satisfies the following equation.

$$(32)^3 \cdot (128)^k = (2048)^{k-7}$$

9. Do you like to play with blocks? I do, because I'm a baby! My grandparents gave me a set of n blocks to play with for my birthday. Each one has a single letter on it and they are all unique except for one pair of letters that are the same. If I can create sixty unique arrangements of the blocks, how many blocks did I receive from my grandparents?

Worcester County Mathematics League
Varsity Meet 1 - October 3, 2018
Team Round Answer Sheet

ANSWERS

1. ____________________

2. ____________________

3. ____________________ times

4. ____________________ yellow socks

5. ____________________ un^2

6. ____________________

7. ____________________ cm

8. ____________________

9. ____________________ blocks

Marlboro, Uxbridge, Bartlett, QSC, Bancroft, St. John's, QSC, Mass Academy, Assabet

Worcester County Mathematics League
Varsity Meet 1 - October 3, 2018
Answer Key

Round 1 - Arithmetic

1. $\frac{19}{20}$ or .95
2. $\frac{5}{3}$ or $1\frac{2}{3}$ or $1.\overline{6}$
3. $\frac{49}{4}$ or $12\frac{1}{4}$ or 12.25

Round 2 - Algebra I

1. 18
2. $(x, y) = \left(\frac{1}{2}, 1\right)$
3. $\frac{60}{13}$ hours or $4\frac{8}{13}$ hours

Round 3 - Set Theory

1. 29 elements
2. 37 seniors
3. 265 integers

Round 4 - Measurement

1. 13 meters
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8. $k = 23$
9. 5 blocks

Round 1 - Arithmetic

1. Simplify the expression $\dfrac{4+3\,(15-(-5))}{8-7(24-32)} \div \dfrac{3\cdot 5-5}{11-\frac{3}{2}}$.

Solution:

$$\frac{4+3\,(15-(-5))}{8-7(24-32)} \div \frac{3\cdot 5-5}{11-\frac{3}{2}} = \frac{64}{64} \div \frac{10}{\frac{19}{2}} = \frac{19}{20} = 0.95$$

2. If $a \ddagger b = \dfrac{a+2b}{2}$ and $a \triangle b = \dfrac{2a-b}{a}$, evaluate $3^{(2\ddagger 1)} \triangle 3$.

Solution:

$$2 \ddagger 1 = \frac{2+2(1)}{2} = 2$$

$$3^2 \triangle 3 = 9 \triangle 3 = \frac{2(9)-(3)}{9} = \frac{5}{3}$$

3. Let $\boxed{A}$ be the arithmetic mean of all odd prime less than A, and let $\textcircled{B}$ be the number of odd prime numbers less than B. Find the value of the following expression.

$$\boxed{\textcircled{\boxed{20}\times\boxed{18}}}$$

Solution: The value of $\boxed{20}$ is

$$\frac{3+5+7+11+13+17+19}{7} = \frac{75}{7}$$

and the value of $\boxed{18}$ is

$$\frac{3+5+7+11+13+17}{6} = \frac{56}{6}.$$

The product of those values is

$$\frac{75}{7}\cdot\frac{56}{6} = 100.$$

There are 24 odd prime numbers less than 100, so $\boxed{\textcircled{100}} = \boxed{24}$ and the value of $\boxed{24}$ is

$$\frac{3+5+7+11+13+17+19+23}{8} = \frac{98}{8} = \frac{49}{4} = 12.25$$

Round 2 - Algebra I

1. If $a < b$, then $3^2 + 4^2 + 5^2 + 12^2 = a^2 + b^2$ is satisfied by only one pair of positive integers a and b. What is the value of $a + b$?

Solution: Since $3^2 + 4^2 = 5^2$ and $5^2 + 12^2 = 13^2$, then

$$(3^2 + 4^2) + (5^2 + 12^2) = 5^2 + 13^2 = a^2 + b^2.$$

Therefore, $a = 5$ and $b = 13$ and $a + b = 18$.

2. Solve for x and y.

$$\begin{cases} \frac{6}{x} - \frac{1}{y} & = 11 \\ \frac{2}{x} + \frac{1}{y} & = 5 \end{cases}$$

Solution: Adding the two equations together yields

$$\frac{8}{x} = 16$$

for which $x = \frac{1}{2}$ is the x-coordinate of the solution. Plugging this back into either equation yields $y = 1$ as the y-coordinate of the solution. The solution is $\left(\frac{1}{2}, 1\right)$.

3. Al and Bob, working together, can do a job in 4 hours. Bob and Carl, working together can do it in 5 hours. All three working together can do it in 3 hours. How long will it take Al and Carl, working as usual, to do the job together without Bob?

Solution: The problem indicates that if a, b, and c are the number of hours it takes Al, Bob, and Carl to each do the job individually, then

$$\begin{cases} \frac{1}{a} & + & \frac{1}{b} & & & = & \frac{1}{4} \\ & & \frac{1}{b} & + & \frac{1}{c} & = & \frac{1}{5} \\ \frac{1}{a} & + & \frac{1}{b} & + & \frac{1}{c} & = & \frac{1}{3} \end{cases}$$

Let x be the number of hours it takes Al and Carl to do the job. Then $\frac{1}{a} + \frac{1}{c} = \frac{1}{x}$. Subtracting the 2nd equation from the 3rd gives us $\frac{1}{a} = \frac{1}{3} - \frac{1}{5} = \frac{2}{15}$, and subtracting the 1st equation from the 3rd gives us $\frac{1}{c} = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$. Therefore,

$$\frac{1}{a} + \frac{1}{c} = \frac{2}{15} + \frac{1}{12} = \frac{39}{180} = \frac{13}{60} = \frac{1}{x}.$$

Therefore, the number of hours it will take Al and Carl to do the job is $x = \frac{60}{13}$ hours, or $4\frac{8}{13}$ hours.

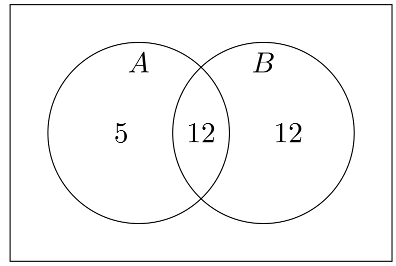
Round 3 - Set Theory

1. Set A contains 17 elements, set B contains 24 elements, and $A \cap B$ contains 12 elements. How many elements are contained in $A \cup B$?

Solution: Since A contains 17 elements and shares 12 elements with B, then there are only 5 elements solely in A. Similarly, there are only 12 elements solely in B. Therefore, there are a total of $5 + 12 + 12 = 29$ elements in total in both A and B. Additionally, the inclusion-exclusion principle for two sets states that

$$|A \cup B| = |A| + |B| - |A \cap B|,$$

which means the number of elements in both sets is $17 + 24 - 12 = 29$.

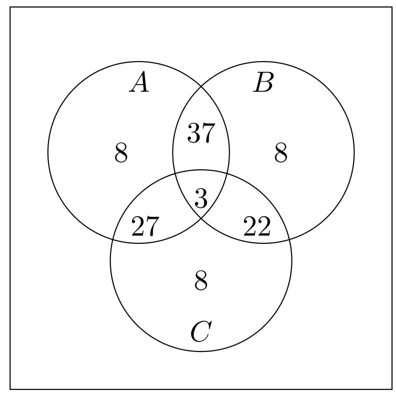


2. The principal of Rosetta Stone High School proudly reported that out of 150 members of the senior class, 75 studied French, 70 studied German, 60 studied Spanish. 40 studied French and German, 30 studied French and Spanish, and 25 studied German and Spanish. If 3 seniors studied all three languages, how many seniors studied none of these languages?

Solution: Let set A represent students studying French, set B represent students studying German, and set C represent students studying Spanish. Given 3 seniors study all three languages, then 37 studied French and German but not Spanish, 27 studied French and Spanish but not German, and 22 studied German and Spanish but not French. Moving forward, this implies that 8 students only take French, 8 others only take German, and 8 others only take Spanish. Adding up these numbers leaves us with 113 students taking languages, meaning $150 - 113 = 37$ students take no languages at all. Additionally, the inclusion-exclusion principle for three sets states that

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|,$$

which means the number of students taking at least one language is $75 + 70 + 60 - 40 - 30 - 25 + 3 = 113$. Subtracting from 150, we arrive at 37 students taking no languages at all.

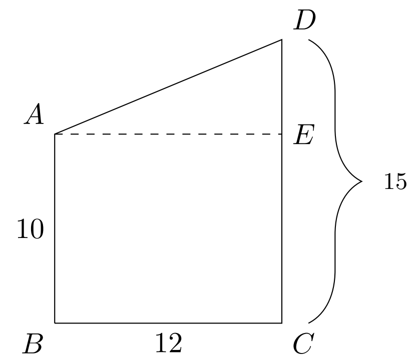


3. How many positive integers less than 1000 have 6, 10, or 15 as factors?

Solution: Let A, B, and C be the sets of positive integers less than 1000 that are divisible by 6, 10, and 15. Then $|A|$, $|B|$, and $|C|$ are the quotients of 1000 divided by 6, 10, and 15 and are $|A| = 166$, $|B| = 99$ (we subtract one since the remainder was 0 and we are only interested in factors less than 1000), and $|C| = 66$. We are trying to find $|A \cup B \cup C|$, which the inclusion-exclusion principle defines as $|A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$. Since $\text{lcm}(6, 10) = \text{lcm}(10, 15) = \text{lcm}(6, 15) = \text{lcm}(6, 10, 15) = 30$, then $|A \cap B| = |B \cap C| = |A \cap C| = |A \cap B \cap C| =$ the quotient of $1000 \div 30 = 33$. Therefore, $|A \cup B \cup C| = 166 + 99 + 66 - 33 - 33 - 33 + 33 = 265$.

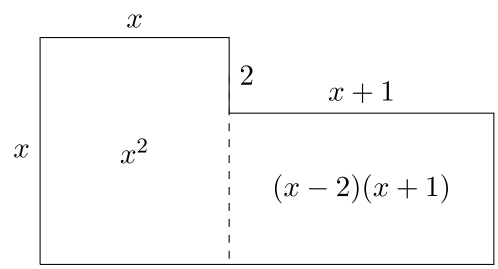
Round 4 - Measurement

1. Two vertical poles, 10 meters high and 15 meters high, stand 12 meters apart. A wire is strung tight between the tops of the poles. Find the length of the wire.



Solution: In quadrilateral $ABCD$, where the wire is represented by $\overline{AD}$, draw $\overline{AE}$ parallel to $\overline{BC}$ to create rectangle $ABCE$ with $AE = 12$, $EC = 10$ and $DE = DC - EC = 15 - 10 = 5$. By the Pythagorean Theorem, $(DE)^2 + (AE)^2 = (AD)^2$, giving us $5^2 + 12^2 = 169 = (AD)^2$ and $AD = 13$ meters.

2. If the area of the figure below is 64 square units, find the perimeter.



Solution: Breaking the figure into a square and a rectangle, the areas must sum to 64.

$$\begin{aligned} x^2 + (x-2)(x+1) &= 64 \\ x^2 + x^2 - 2x + x - 2 &= 64 \\ 2x^2 - x - 66 &= 0 \\ (2x+11)(x-6) &= 0 \end{aligned}$$

The solutions to the equation are -5.5 and 6, and since x must be positive, $x = 6$ and the perimeter is $6 + 6 + 6 + 7 + 7 + 6 = 38$.

3. A regular square pyramid with equilateral triangles as lateral faces has volume $\dfrac{1372\sqrt{2}}{3}$ cubic meters. What is the length of a lateral edge?

Solution: Let x be the length of each edge of the pyramid. The volume of the pyramid is $V = \frac{1}{3}Bh = \frac{1}{3}(x^2)h$. To find the height, consider the right triangle created by the slant height of the pyramid (hypotenuse), the height of the pyramid (leg), and the line connecting the midpoint of a base edge with the center of the base (leg). The slant height of the pyramid is the height of an equilateral triangle with edge length x which is $\frac{\sqrt{3}}{2}x$, and the line connecting the baseedge with the center of the base is just half a side length and is $\frac{1}{2}x$. To find the height,

$$\begin{aligned} \left(\tfrac{1}{2}x\right)^2 + h^2 &= \left(\tfrac{\sqrt{3}}{2}x\right)^2 \\ h^2 &= \tfrac{3}{4}x^2 - \tfrac{1}{4}x^2 \\ h &= \tfrac{x}{\sqrt{2}} \end{aligned}$$

Since we now have $V = \frac{1}{3}(x^2)\left(\frac{x}{\sqrt{2}}\right) = \frac{x^3}{3\sqrt{2}}$, then setting $\frac{1372\sqrt{2}}{3} = \frac{x^3}{3\sqrt{2}}$ yields $x^3 = 2744$ and $x = 14$ meters.

Round 5 - Polynomial Equations

1. Find all values of x which satisfy $4^{x^3+5x^2-6x} = 1$.

Solution: Since $4^0 = 1$, we must find solutions to the equation $x^3 + 5x^2 - 6x = 0$

$$\begin{aligned} x^3 + 5x^2 - 6x &= 0 \\ x(x^2 + 5x - 6) &= 0 \\ x(x-1)(x+6) &= 0 \end{aligned}$$

The solutions are $0, 1, -6$.

2. Given that 3 and $-\frac{1}{2}$ are solutions of $2x^4 - x^3 - 3x^2 - 31x - 15 = 0$, determine the other roots.

Solution: Using synthetic division, we divide $2x^4 - x^3 - 3x^2 - 31x - 15$ by $x - 3$ and the resulting quotient by $x - \frac{1}{2}$:

$$\begin{array}{r|rrrrr} & 2 & -1 & -3 & -31 & -15 \\ 3 & & 6 & 15 & 36 & 15 \\ \hline & 2 & 5 & 12 & 5 & 0 \end{array} \qquad \begin{array}{r|rrrr} & 2 & 5 & 12 & 5 \\ -\frac{1}{2} & & -1 & -2 & -5 \\ \hline & 2 & 4 & 10 & 0 \end{array}$$

We can now rewrite the original equation as $(2)(x-3)(x+\frac{1}{2})(x^2+2x+5) = 0$. Finding the solutions for the quadratic will give us our last two roots. By the quadratic formula,

$$x = \frac{-2 \pm \sqrt{4-20}}{2} = -1 \pm \frac{4i}{2} = -1 \pm 2i.$$

The last two roots are $-1 + 2i$ and $-1 - 2i$.

3. Determine the square roots of $5 - 12i$.

Solution: Let a and b be real numbers and $a+bi$ be a square root of $5-12i$. Then $(a+bi)^2 = a^2+2abi+b^2i^2 = (a^2-b^2)+2abi = 5-12i$ and we have two separate equations:

$$a^2 - b^2 = 5 \text{ and } ab = -6.$$

Substituting $a = -\frac{6}{b}$ into the first equation, multiplying through by b^2, and factoring, we find

$$\frac{36}{b^2} - b^2 = 5 \rightarrow 36 - b^4 = 5b^2 \rightarrow b^4 + 5b^2 - 36 = 0 \rightarrow (b^2+9)(b^2-4) = 0 \rightarrow b^2 = -9 or b^2 = 4.$$

Since b is real, $b = 2$ or $b = -2$. Inserting into the second equation, if $b = 2$ then $a = -3$ and if $b = -2$ then $a = 3$. The two roots are $-3 + 2i$ and $3 - 2i$.

Team Round

1. If $x\&y = xy+1$ and $x\#y = x+y+1$, what is the value of $4\&[(6\#8)\#(3\&5)]$?

Solution: Following the order of operations,

$$4\&[(6\#8)\#(3\&5)]$$
$$4\&[(15)\#(16)]$$
$$4\&[32]$$
$$129$$

2. If

$$\begin{cases} x+y+z &= 9 \\ x+2y+4z &= 15 \\ x+3y+9z &= 23 \end{cases}$$

determine the value of xyz.

Solution: Subtracting the first equation from both the second and third equations leaves the bottom two equations as a single system:

$$\begin{cases} y+3z &= 6 \\ 2y+8z &= 14 \end{cases}$$

Subtracting twice the first equation from the second leaves the equation $2z = 2$ which sets $z = 1$. Plugging this back in to the two-equation system gives $y = 3$. Plugging these back into the original first equation gives $x = 5$. The product $xyz = 15$.

3. How many times does the digit two appear if you write out the integers between 100 and 300?

Solution: There are one hundred numbers that begin with the number 2 (100 overall). There are ten numbers per hundred whose number in the tens place is two (10 per hundred) and ten numbers per hundred whose number in the ones place is two (10 per hundred). This means the digit two appears 140 times.

4. Paul's sock drawer is filled with 26 socks that are either red, yellow, or blue. The probability that Paul pulls a red sock, a yellow sock, and a blue sock (in that order) is $\frac{1}{40}$. If red socks make up half of his drawer and he has more yellow socks than blue socks, how many yellow socks does Paul have?

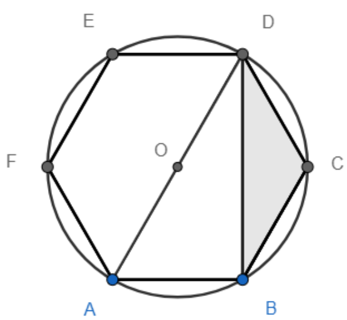
Solution: Let n be the number of socks that Paul has, and let R, Y, and B be the number of red, yellow, and blue socks that Paul has, respectively. Then

$$\frac{1}{40} = \left(\frac{R}{26}\right)\left(\frac{Y}{25}\right)\left(\frac{B}{24}\right) = \left(\frac{1}{2}\right)\left(\frac{Y}{25}\right)\left(\frac{B}{24}\right)$$
$$\frac{1}{20} = \left(\frac{Y}{25}\right)\left(\frac{B}{24}\right)$$
$$600 = 20YB$$
$$30 = YB.$$

Since $Y + B = 13$, then solving this system (or by guess and check) yields 10 yellow socks and 3 blue socks. Why Paul only has three blue socks, I have no idea and take no responsibility for what comments the students may have.

5. Regular hexagon ABCDEF is inscribed in circle O whose area is 4π. Find the area of ΔBCD.

Solution: Circle O, wi
hexagon $ABCDEF$ has
to a vertex must also be
of six equilateral triangle
and height as triangle BC
with side length 2 has ar



6. Find all complex numbers whose negative reciprocal is equal to its square.

Solution: Let x be a complex number of the form $a + bi$. By the description of the problem, we can write

$$-\frac{1}{x} = x^2$$

which follows with

$$x^3 = -1$$
$$x^3 + 1 = 0$$
$$(x+1)(x^2 - x + 1) = 0$$

The first factor indicates a solution to be $x = -1$. The second factor, by the Quadratic Formula, indicates the two other solutions to be $x = \frac{1 \pm \sqrt{1-(4)(1)(1)}}{2(1)} = \frac{1 \pm \sqrt{-3}}{2} = \frac{1}{2} + \frac{\sqrt{3}}{2}i$ or $\frac{1}{2} - \frac{\sqrt{3}}{2}i$.

7. Lynn has just filled up her cone-shaped water bottle, which has a base radius of 6 cm and a height of 10 cm. She drinks exactly half of her water and sets the bottle back down onto the table base-down. Determine the height of the water level in the bottle.

Solution: The volume of a cone with base radius 6 cm and height of 10 cm is

$$V_{cone} = \frac{1}{3}\pi(6)^2(10) = 120\pi.$$

Half of this volume would be 60π cm^3, so this is the amount of water left in the water bottle when she sets the cone back down. The 60π cm^3 of water comes to rest at the bottom of the cone, and there is an air pocket above in the shape of a cone (we will call this the air-cone) that also has a volume of 60π cm^3. Finding the height of this air-cone will allow us to find the height of the water left in the cone. Since the full cone and the air-cone are similar, the air-cone's radius and height are at a ratio of 6:10 or 3:5 letting us determine that $\frac{h}{r} = \frac{10}{6} \Rightarrow r = \frac{3h}{5}$. Therefore,

$$V_{air-cone} = 60\pi = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\frac{3h}{5}\right)^2 h = \frac{3}{25}\pi h^3.$$

Solving for h, the height of the air-cone, gives us $h = \sqrt[3]{500}$. This means the height of the water left in the bottle is at a height of $10 - \sqrt[3]{500}$ cm (approximately 2 cm).

8. Determine the value of k that satisfies the following equation: $(32)^3 \cdot (128)^k = (2048)^{k-7}$

Solution: Rewrite the equation with bases of 2.

$$(2^5)^3 \cdot (2^7)^k = (2^{11})^{k-7} \qquad \Rightarrow \qquad 2^{15} \cdot 2^{7k} = 2^{11(k-7)} \qquad \Rightarrow \qquad 2^{15+7k} = 2^{11k-77}$$

Setting the exponents equal to each other yields an equation for which $k = 23$ is the solution.

9. Do you like to play with blocks? I do, because I'm a baby! My grandparents gave me a set of n blocks to play with for my birthday. Each one has a single letter on it and they are all unique except for one pair of letters that are the same. If I can create sixty unique arrangements of the blocks, how many blocks did I receive from my grandparents?

Solution: If there are n unique blocks there are $n!$ unique arrangements, but in this case there are 2 blocks that are identical, meaning there are $\frac{n!}{2!}$ unique arrangements. Knowing that there are 60 unique arrangements of the baby's blocks, $n!$ must be equal to 120, so $n = 5$ blocks.